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SPLITTING OFF TORI AND THE EVALUATION SUBGROUP OF THE FUNDAMENTAL GROUP

BY

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ABSTRACT

Given a space X what is the largest torus T^n such that X is homotopy equivalent to $Y \times T^n$? We find the answer depends on a simple property of the evaluation subgroup of the fundamental group, $G_1(X)$. As corollaries we have the Splitting theorem of Conner and Raymond and the fact that the dimension of X must be greater than the rank of $G_1(X)$.

In this paper all spaces are CW complexes. All spaces are assumed to be connected. We say that X has *toral number n* if n is the largest integer such that X is homotopy equivalent to $Y \times T^n$ for some space Y. Note that Y is necessarily homotopically equivalent to a covering space of X.

Let G be a subgroup of $\pi_1(X)$. Define the *Hurewicz rank of G* as follows. Consider the image of G under the Hurewicz homomorphism h in the homology group. Then $h(G)$ may contain free summands of $H_1(X)$. We say the *Hurewicz rank of G* is the maximum rank of these free summands. If there is no free summand in $h(G)$ then we say the Hurewicz rank of G is zero and if there is no maximum we say the Hurewicz rank of G is infinite.

THEOREM. *The toral number of X is equal to the Hurewicz rank of $G_1(X)$.*

COROLLARY 1. *There exists a finite covering $\tilde{X}$ of X whose toral number is greater than or equal to the rank of $h(G_1(X))$.*

COROLLARY 2. *The dimension of X is greater than or equal to the rank of $G_1(X)$.*

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COROLLARY 3. *If F is dominated by a finite complex and is the fibre of a fibration with simply connected total space, then F is homotopically equivalent to a finite complex except possibly when F has a nontrivial finite fundamental group.*

COROLLARY 4 (Conner–Raymond Splitting Theorem [C–R]). *If a torus T^k acts on X so that evaluation at a point gives a map $\omega: T^k \to X$ so that $\omega_*(H_1(T^k))$ is a direct summand of $H_1(X)$ of rank k, then X is equivariantly homeomorphic to $T^k \times Y$ for some space Y where T^k acts on the product by $g(h, y) = (gh, y)$.*

This paper was inspired by a question of Haefliger [H–S], to which Corollary 3 is a simplified answer. It owes its existance to a key observation of Siebenmann. The main result is related to a result of Wolfgang Lück. The relationship will be discussed in the last section. John Oprea (*A homotopical Conner–Raymond theorem and a question of Gottlieb*, Can. Math. Bull., to appear) has independently discovered a version of the main theorem and its relation to the Conner–Raymond splitting theorem.

1. The evaluation subgroup

Here we recall some facts concerning the evaluation subgroup $G_1(X)$ which hereafter we will denote by $G(X)$. These results may be found in [G1]. There are three equivalent definitions of the evaluation subgroup:

$G(X) = \omega_*(\pi_1(X^X, 1_X))$ = *the group of traces of cyclic homotopies* = *the group of deck transformations of the universal covering which are homotopic to the identity by a homotopy which commutes with all Deck transformations.*

Here we let X^X denote the space of maps of X to itself and ω is the evaluation map at a base point. A *cyclic homotopy* is a homotopy h_t so that $h_0 = h_1 = 1_X$. Then the *trace* is the element in the fundamental group of X corresponding to the loop $h_t(x_0)$ where x_0 is the base point.

Much is known about $G(X)$. Here are some properties.

(1) $G(X)$ is a homotopy invariant.

(2) $G(X \times Y) = G(X) \times G(Y) \subset \pi_1(X) \times \pi_1(Y)$.

(3) $G(X) = \pi_1(X)$ if X is an H-space.

(4) $G(X) = 0$ if X is a finite complex whose Euler–Poincaré number $\chi(X) = 0$.

(5) $G(X)$ is contained in the center of $\pi_1(X)$.

(6) $G(K(\pi, 1))$ = center of π.

(7) $G(X)$ contains the image of every connecting homomorphism $d: \pi_2(B) \to \pi_1(X)$ which arises from a fibration $X \to E \to B$.

(8) $G(X) \cap H \subset G(\tilde{X})$ where $\tilde{X}$ is the covering space associated to the subgroup H of $\pi_1(X)$.

Let us combine, for example, statement 3 that $G(X) = \pi_1(X)$ if X is an H-space with some of the corollaries. We immediately read out that an H-space is the product of a torus, whose dimension is equal to the rank of its fundamental group, with an H-space with torsion fundamental group; and that if the H-space were finitely dominated it would be homotopy equivalent to a finite complex unless the fundamental group were non-trivial and finite. This last part is a theorem of Mislin [M]. The first part is well known of course, especially in the classification of Lie groups.

2. Proof of the Theorem

First consider a map $f: X \to K(\pi, 1)$. Supose that $f_*: \pi_1(X) \to \pi_1(K(\pi, 1)) = \pi$ is onto. Then the homotopic theoretic fibre F of f must be connected and is a regular covering space of X corresponding to the subgroup kernel f_* of $\pi_1(X)$. In addition, the action of π on $\pi_1(F) =$ kernel f_* is induced by the action of the group of Deck transformations of F acting on F. This group of Deck transformations is isomorphic to $\pi_1(X)/\text{kernel } f_* = \pi$.

LEMMA (Siebenmann). *Suppose that $f: X \to S^1$ has homotopy theoretic fibre F. If $\pi_1(S^1)$ acts homotopy trivially on F then X must be homotopy equivalent to $S^1 \times F$.*

There are two ways to see this fact. The hypothesis that $\pi_1(S^1)$ acts homotopy trivially on F means that the clutching map of the fibration, $g: F \to F$, is homotopic to the identity. Hence the mapping torus M_g is homotopy equivalent to $S^1 \times F$. But M_g is homotopy equivalent to the total space X. A second proof takes advantage of the fact that there is a universal fibration $F \to E_\infty \to B_\infty$ with fibre F so that $F \to X \to S^1$ is a pullback. Now $\pi_1(B_\infty)$ is isomorphic to the group of self homotopy equivalences and the classifying map $S^1 \to B_\infty$ represents the element of $\pi_1(B_\infty)$ corresponding to g. So the pullback must be the trivial fibration so its total space X is homotopy equivalent to $S^1 \times F$.

Now we want to use the lemma and induction to split off another circle. Unfortunately induction does not work since the fact that a Deck transformation is homotopic to the identity on some covering space does not imply that its induced version is homotopic to the identity on a covering space corresponding to a smaller subgroup. However, if the Deck transformation were equivariantly homotopic to the identity, then its induced versions would be homotopic to the identity on every subspace. That is why the evaluation subgroup plays the key role in splitting off tori.

So denote the Hurewicz rank of $G(X)$ by k and the toral number of X by l. We have k elements $\alpha_i \in G(X)$ such that $h(\alpha_i)$ are independent generators of $H_1(X)$. Now there is a map $X \to S^1$ which takes $h(\alpha_1)$ to the generator of $H_1(S^1)$ since $h(\alpha_1)$ is a generator and is not torsion. The clutching map of the fibration induced by this map is the Deck transformation of the covering space X_1 associated with this map (noting X_1 is homotopy equivalent to the homotopy fibre). Now the Deck transformation is the one induced by α_1. Since α_1 is an element in $G(X)$, the Deck transformation is homotopic to the identity and so we can apply the lemma so that $X = S^1 \times X_1$. Now we want to split off an S^1 from X_1. To do this we will use α_2. Now α_2 is an element of $G(X) = G(S^1 \times X_1) = G(S^1) \times G(X_1)$. We can regard α_1 as the generator of $G(S^1)$ and we regard α_2 as a generator of $G(X_1)$. In addition $h(\alpha_2)$ is a torsion free generator of $H_1(X) = H_1(S^1) \oplus H_1(X_1)$ and so we can also think of it as a generator of $H_1(X_1)$. Thus we can apply the lemma to X_1 and get $X_1 = S^1 \times S^1 \times X_2$. We continue in this way until we exhaust all k of the α_i's and split off a k torus. Hence $l \geqq k$.

To show that $l \leqq k$, we consider $T^l \times X$. Then $G(T^l \times X) = G(T^l) \oplus G(X)$ and so $h(G(T^l \times X)) = hG(T^l) \oplus hG(X)$. Also

$$hG(T^l) = h(\pi_1(T^l)) = H_1(T^l) \subset H_1(T^l) \oplus H_1(X),$$

and we see that the toral number is smaller than k, hence the theorem is proved.

3. Proof of the corollaries

Proof of Corollary 1. Suppose that F is a free summand of $H_1(X)$. Then there is a homomorphism from $\pi_1(X) \to F$ factoring through the Hurewicz homomorphism h. If F has rank k, then it is the fundamental group of the k-torus T^k. Hence there is a map $f: X \to T^k$ which induces the above homomorphism. We will choose F to be a free direct summand of smallest rank which contains the free part of $h(G(X))$. Then $H = hG(X) \cap F$ is a subgroup of F of finite order. Hence the covering of T^k corresponding to H is a finite cover and hence a torus of dimension k also. Let $\tilde{X} \to T^k$ be the pullback of $X \to T^k$ by the covering map $T^k \to T^k$. Then $\tilde{X}$ is the required finite covering of X. We observe that $\tilde{X} \to T^k$ maps $H = hG(X) \cap F$ onto $\pi_1(T^k)$. Now H has the same rank as $hG(X)$ and $H \subset G(\tilde{X})$ by property (8). So the theorem implies that we can split T^k off $\tilde{X}$.

PROOF OF COROLLARY 2. Choose the covering $\tilde{X}$ of X corresponding to the subgroup $G(X)$. Then $G(X) = G(\tilde{X})$, so by the theorem one can split off a torus from $\tilde{X}$ whose dimension equals

$$\operatorname{rank} hG(\tilde{X}) = \operatorname{rank} G(\tilde{X}) = \operatorname{rank} G(X).$$

Now the dimension of X equals the dimension of its covering space $\tilde{X}$ which is not smaller than the torus.

PROOF OF COROLLARY 3. By property (7) and the homotopy exact sequence of the fibration, $G(F) = \pi_1(F)$. Hence if $\pi_1(F)$ is not finite we see that F has nonzero toral number. Then we can regard F as a product of a circle with a covering space, $S^1 \times \tilde{F}$. So $\tilde{F}$ is dominated by a finite complex since F is. But the product of a finitely dominated complex with a circle is a finite complex.

PROOF OF COROLLARY 4. The map $\omega : T^k \to X$ factors through $\omega : X^X \to X$, so the Hurewicz rank of $G(X)$ is greater than or equal to k. The isomorphorphism $H_1(T^k) \to \omega_*(H_1(T^k))$ induces a homeomorphism

$$T^k \to \text{orbit} \to X = T^k \times Y \to T^k.$$

Since $T^k \to$ orbit is a quotient map of T^k by the isotropy subgroup, it must be a homeomorphism since $H_1(T^k)$ is mapped onto $H_1(T^k)$. Thus T^k acts freely on X, since it is homeomorphic to every orbit. So $X \to X/T^k$ must be a principal bundle with fibre T^k. Since X retracts onto a fibre, it is a trivial principal bundle and so it is a product with action as stated in the corollary.

4. Haefliger's question

Suppose $F \to E \to B$ is a fibration with the additional conditions that E and B are simply connected and have finite type. We suppose that B contains a torus T^K and that the restriction of the fibration to T^K is homotopically a map $M \to T^K$ where M is a closed manifold. Must F satisfy Poincaré Duality?

If F is a finite complex, then we know that F is a Poincaré Duality space in the sense of Spivak. This follows from [G2] because for any fibration whose fibre, base, and total space are finite complexes, the total space is a Poincaré Duality space if and only if the fibre and the base are Poincaré Duality spaces.

If F is merely dominated by a finite complex we can show that it must be dominated by a Poincaré Duality complex by the following argument. First we observe that $S^1 \times F$ is homotopic to a finite complex. Then we observe that $S^1 \times F \to S^1 \times M \to T^K$ is a fibration of finite complexes and $S^1 \times M$ satisfies Poincaré Duality.

Now in Haefliger's situation we can show that F must always be dominated by a finite complex. A CW complex is dominated by a finite complex if and only if it is homotopy equivalent to a complex of finite dimension and also to a complex of finite type. So we must show that F is both of finite type and finite dimension.

Now F must be finite dimensional since the fibration $F \to M \to T^K$ implies that F is homotopy equivalent to a covering of the closed manifold M. Since B and E are simply connected of finite type we see that F must be of finite type. This follows from the following argument. Since B is simply connected and of finite type, it must have finitely generated homotopy groups. Hence its loop space ΩB has finitely generated homotopy groups. So the universal covering of ΩB has finitely generated homotopy groups and so it is of finite type. Now the fibration $\Omega B \to K(\pi_2(B), 1)$ has the universal covering of ΩB as its fibre. Since the fibre and base space are of finite type, the total space ΩB must have finite type. Now the fibration $\Omega B \to F \to E$ implies that F has finite type for the same reason.

Under what conditions is F actually homotopy equivalent to a finite complex? When $\pi_1(F)$ is trivial or infinite by Corollary 3.

5. Lück rank

In Proposition 4.3 of [L], Lück found a necessary and sufficient condition so that a CW-complex X splits as a product of a circle and a complex Y. His proof may be extended without change to give the following characterisation of toral number.

THEOREM (Lück). *Let X be a CW-complex. There exists a CW-complex Y with X homotopy equivalent to $Y \times T^k$ if and only if $\pi_1(X)$ can be written as $G \times F$ where F is a free abelian group of rank k such that $G_1(X)$ contains F.*

Let us say that $G_1(X)$ has *Lück rank k* if it contains a free abelian direct product factor of $\pi_1(X)$ of rank k. Then Lück's theorem states that the toral number of X is equal to the Lück rank of $G_1(X)$. Now the Hurewicz rank of

$G_1(X)$ is obviously greater than or equal to the Lück rank. But since they both characterize the toral number, they must be equal.

In fact the Lück rank and the Hurewicz rank should be equal for any central subgroup. However, for our applications we need the more tractible formulation of Hurewicz rank.

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